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# Dilaton Gravity Coupled to a Nonlinear Sigma Model in $2+\epsilon$ Dimensions

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**Abstract**

Quantum theory of dilaton gravity coupled to a nonlinear sigma model with a maximally symmetric target space is studied in $2+\epsilon$ dimensions. The ultraviolet stable fixed point for the curvature of the nonlinear sigma model demands a new fixed point theory for the dilaton coupling function. The fixed point of the dilaton coupling is a saddle point similarly to the previous case of the flat target space.

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It has been proposed sometime ago [1] that the quantum theory of gravity can consistently be defined by an analytic continuation from lower dimensions, similarly to the successful application for nonlinear sigma models [2]. The power-counting arguments necessitate an expansion of the quantum gravity around two dimensions. On the other hand, the usual Einstein action becomes a topological invariant precisely at two dimensions. The clash between the singular nature of the Einstein action at two dimensions and the necessity for an expansion around two dimensions is the fundamental origin of a problem afflicting the $(2+\epsilon)$-dimensional approach to quantum gravity. If one takes the conformal gauge, one finds that the propagator for the Liouville field becomes singular and an oversubtraction is needed at the one-loop order [3]. The singular propagator together with the oversubtraction leads to the nonrenormalizable divergences at the two-loop order [4], [5].

To avoid the clash between the singular action and the expansion around two dimensions, it has been proposed to abandon the general coordinate invariance and to use only volume-preserving diffeomorphisms as the fundamental principle [6]. On the other hand, in order to maintain the general coordinate invariance, we have proposed an alternative approach in a preceding paper to solve the difficulty: the dilaton gravity can be used to define the higher dimensional quantum theory of Einstein gravity [7]. We have observed that the dilaton gravity action is equivalent to the Einstein action with an additional free scalar field, except in two spacetime dimensions. Most importantly, the dilaton gravity action is smooth around two dimensions. As a result, the Liouville-dilaton propagator becomes nonsingular even in two dimensions. Therefore the nonrenormalizable divergences arising from the singular Liouville propagator should be absent in our dilaton gravity theory. We have explicitly studied the case of $N$ free massless scalar fields as matter fields interacting with the dilaton-gravity system. We have obtained divergences and beta functions to one-loop which exhibit a nontrivial fixed point. The fixed point is ultraviolet stable for the gravitational coupling constant $G$, if $\epsilon > 0$ and $N < 24$, although it is not ultraviolet stable for the strength of the dilaton coupling function [7]. We have also found that the fixed point theory can be transformed to an action of the usual CGHS type [8]. The $(2+\epsilon)$-dimensional gravity with or without dilaton have been applied to other interesting situations [9] – [12].

The results on the dilaton gravity, however, may depend on the choice of possible interactions among matter fields. Therefore it is worthwhile to study the renormalization group properties of the dilaton gravity further using models with interacting

matter fields.

The purpose of this letter is to study the dilaton gravity coupled to a nontrivially interacting scalar fields. For simplicity, we consider a nonlinear sigma model with a maximally symmetric target space as an interacting matter fields. We find a nontrivial fixed point which is ultraviolet stable in the direction of both the gravitational coupling constant $G$ and the curvature of the maximally symmetric target space $\alpha'$, but is a saddle point in the direction of the dilaton coupling function. It turns out that the dilaton gravity at the fixed point can be transformed to a free dilaton field embedded into the Einstein gravity, if fields are redefined with a local Weyl rescaling which is singular as $\epsilon \to 0$. Since the transformation is singular in the limit of two dimensions, such a transformation is not allowed in our $(2+\epsilon)$-dimensional approach. Therefore the fixed point theory cannot be regarded as a free dilaton gravity theory. The necessity of fine tuning of the dilaton coupling function is a feature which is common to our previous result on the free massless scalar fields [7]. On the other hand, the new fixed point of our nonlinear sigma model with the maximally symmetric target space corresponds to an ultraviolet stable nontrivial fixed point ($\alpha' \neq 0$) for the curvature of the target space. This fixed point does not reduce to our previous result for the free massless scalar fields which corresponds to an infrared stable trivial fixed point ($\alpha' = 0$) for the curvature. Note that both of them require a fine-tuning of the dilaton coupling function.

Let us consider $N+1$ spinless fields $\phi, X^j$ $(j = 1, \cdots, N)$. We require the power-counting renormalizability in two-dimensional limit and find the most general action as a nonlinear sigma model (including the dilaton $\phi$) coupled to gravity as

$$S = \mu^\epsilon \int d^d x \sqrt{-g} \Big[ \frac{1}{16\pi G} R^{(d)} L(\phi, X) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi G_{\phi\phi}(\phi, X) \\ - g^{\mu\nu} \partial_\mu \phi \partial_\nu X^j G_{\phi j}(\phi, X) - \frac{1}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j G_{ij}(\phi, X) \Big], \tag{1}$$

which contains four arbitrary functions $L$, $G_{\phi\phi}$, $G_{\phi j}$, $G_{ij}$ of $\phi$ and $X^i$. Here, $d = 2+\epsilon$, and $G$ and $\mu$ are the renormalized gravitational constant and the renormalization scale respectively. In this paper, we consider a nonlinear sigma model for the interacting matter fields $X^i$ to form the maximally symmetric target space in $N$ dimensions as the simplest target space (of $X^i$) which has a nontrivial geometry. The maximal symmetry requires the coupling functions to be

$$L(\phi, X) = L(\phi), \qquad G_{\phi j}(\phi, X) = 0,$$

$$G_{\phi\phi}(\phi, X) = \Psi(\phi), \quad G_{ij}(\phi, X) = \frac{1}{2\pi\alpha'}\bar{G}_{ij}(X)\,\mathrm{e}^{-2\Phi(\phi)}, \tag{2}$$

where $\alpha'$ is proportional to the curvature of the target space. The metric of the maximally symmetric space $\bar{G}_{ij}$ satisfies

$$\bar{R}_{ijkl} = k\left(\bar{G}_{ik}\bar{G}_{jl} - \bar{G}_{il}\bar{G}_{jk}\right), \tag{3}$$

where the parameter $k$ takes its value on $\{0, \pm 1\}$. The symmetry of the space of the matter fields is $O(N+1)$ for $k = +1$, $O(N,1)$ for $k = -1$ and the $N$-dimensional Euclidean group for $k = 0$ respectively. The $k = 0$ case has been studied in our previous paper [7]. In this paper we study the cases $k = \pm 1$. To fix the meaning of the curvature of the target space, we choose $\Phi(0) = 0$.

Similarly to the previous case of the dilaton gravity with free massless scalar fields [7], we have two kinds of freedom to redefine the fields. The first one is the local Weyl rescaling of the metric $g_{\mu\nu} \to \mathrm{e}^{-2\Lambda(\phi)}g_{\mu\nu}$ with a function $\Lambda$ of the dilaton $\phi$. This can be used to fix one of the functions $L, \Psi$, or $\Phi$. Since the function $\Psi$ changes by terms of order $\epsilon^0$ by the local Weyl rescaling, whereas $L, \Phi$ change only by terms of order $\epsilon$, we choose to fix the function $\Psi$ by means of the local Weyl rescaling. We can find the local Weyl rescaling $\Lambda$ which transforms a generic model to the model with $\Psi(\phi) = 0$, and is finite as we let $\epsilon \to 0$. Let us note that this choice fixes only $\Lambda'(\phi)$, namely the nonzero modes of $\Lambda(\phi)$. The second field redefinition is an arbitrary field redefinition of the dilaton $\phi$ with a function $f$ of the dilaton $\phi \to f(\phi)$. We can use it to fix the form of the function $L(\phi)$, which we choose $L(\phi) = \exp(-2\phi)$ as the standard form. This fixes only nonzero modes of $f(\phi)$. The zero modes of $\Lambda(\phi)$ and $f(\phi)$ are used to fix coefficients of two reference operators, analogously to the case of free massless scalar fields [7]. With these choices of $\Lambda(\phi)$ and $f(\phi)$, we obtain the following standard form of the action of dilaton gravity coupled to the nonlinear sigma model

$$S = \mu^{\epsilon}\int d^d x\sqrt{-g}\Big[\frac{1}{16\pi G}R^{(d)}\,\mathrm{e}^{-2\phi} - \frac{1}{4\pi\alpha'}g^{\mu\nu}\partial_\mu X^i\partial_\nu X^j\bar{G}_{ij}(X)\,\mathrm{e}^{-2\Phi(\phi)}\Big], \tag{4}$$

where $\Phi(0) = 0$.

We use the background field method [13] to compute one-loop divergences. Let us introduce the background metric $\hat{g}_{\mu\nu}$ and decompose the metric $g_{\mu\nu}$ into the

traceless field $h_{\mu\nu}$ and the Liouville field $\rho$

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}\,\mathrm{e}^{-2\rho} = \hat{g}_{\mu\lambda}(\,\mathrm{e}^{\kappa h})^{\lambda}{}_{\nu}\,\mathrm{e}^{-2\rho}, \tag{5}$$

where $\kappa^2 = \frac{16\pi G}{\mu^\epsilon}$. The action (4) becomes

$$S = \mu^\epsilon \int d^dx \sqrt{-\tilde{g}}\,\mathrm{e}^{-\epsilon\rho}\Big[\frac{1}{16\pi G}\Big(\tilde{R}^{(d)}\,\mathrm{e}^{-2\phi} + \epsilon(\epsilon+1)\tilde{g}^{\mu\nu}\partial_\mu\rho\partial_\nu\rho\,\mathrm{e}^{-2\phi}$$
$$+4(\epsilon+1)\tilde{g}^{\mu\nu}\partial_\mu\rho\partial_\nu\phi\,\mathrm{e}^{-2\phi}\Big) - \frac{1}{4\pi\alpha'}\tilde{g}^{\mu\nu}\partial_\mu X^i\partial_\nu X^j\bar{G}_{ij}(X)\,\mathrm{e}^{-2\Phi(\phi)}\Big]. \tag{6}$$

At this point we emphasize the following point for the nonsingular nature of our dilaton gravity theory. We can readily read off the propagator for the Liouville-dilaton system from the above action and find that they have smooth two-dimensional limit $\epsilon \to 0$. More generally, even if we allow the most general coupling between dilaton and other matter fields including $G_{\phi j}(\phi, X) \neq 0$ in Eq. (2), we find that the propagator of all fields are nonsingular in the limit of two dimensions provided the matter kinetic terms are nonsingular $\det G_{ij} \neq 0$. Therefore the nonsingular propagator and the smoothness of the dilaton gravity theory is a generic feature irrespective of the details of possible couplings among matter and dilaton fields. This point is crucial [7] to eliminate nonrenormalizable divergences observed in Refs. [4], [5].

Let us make use of our previous result for one-loop calculation for a general action [7] involving the Liouville field together with the dilaton and matter fields. Combining all the scalar fields and the Liouville field into $Y^I = (\rho, \phi, X^i)$, where the index runs $I = (\rho, \phi, i)$, we obtain a kind of nonlinear sigma model using the metric $\tilde{g}_{\mu\nu} = \hat{g}_{\mu\lambda}(\,\mathrm{e}^{\kappa h})^{\lambda}{}_{\nu}$ with the Liouville field $\rho$ removed

$$S = \frac{\mu^\epsilon}{16\pi G}\int d^dx\sqrt{-\tilde{g}}\left[\tilde{R}^{(d)}L(Y) - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu Y^I\partial_\nu Y^J G_{IJ}(Y)\right], \tag{7}$$

where the curvature for the metric $\tilde{g}_{\mu\nu}$ is denoted as $\tilde{R}^{(d)}$. The functions $L(\phi)$ and $G_{IJ}$ in our specific model can be read off from Eq. (6). The gauge fixing term and the ghost action are given in terms of the gauge fixing function $F_\alpha$ [7] as

$$S_{\mathrm{GF+FP}} = \int d^dx\,\delta_{\mathrm{B}}(-ib^\alpha F_\alpha),$$
$$F_\alpha = \sqrt{-\hat{g}}\hat{L}\left[\hat{D}^\beta h_{\beta\alpha} - \frac{1}{\kappa}\partial_\alpha\left(\frac{L(Y)}{\hat{L}}\right) + \frac{1}{2}B_\alpha\right], \tag{8}$$

where $\delta_{\rm B}$ is the BRST transformation and the fields with a hat are background. The one-loop divergences in the effective action of the general model (7) with the gauge fixing term and the ghost action (8) is given by [7]

$$\Gamma_{\rm div} = \int d^d x \sqrt{-\hat{g}} \Big[ \frac{24-N}{24\pi\epsilon} \hat{R}^{(d)} - \frac{1}{4\pi\epsilon} \hat{g}^{\mu\nu} \partial_\mu \hat{Y}^I \partial_\nu \hat{Y}^J \left( \hat{R}_{IJ} + \partial_I \ln \hat{L} \partial_J \ln \hat{L} \right) \Big]. \quad (9)$$

Let us apply this result to our model (4). By computing the curvature in the $Y^I = (\rho, \phi, X^i)$ space from the metric in Eq. (6), we obtain the one-loop divergence of the dilaton gravity coupled to the maximally symmetric nonlinear sigma model. The counter terms can be summarized with three types of coefficients $A, B, C$

$$\begin{aligned} S_{\rm counter} &= -\mu^\epsilon \int d^d x \sqrt{-g} \Big[ R^{(d)} A(\phi) + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi B(\phi) \\ &\quad - \frac{1}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i \bar{G}_{ij}(X) C(\phi) \Big]. \end{aligned} \quad (10)$$

We find that these coefficients are given at one-loop order by

$$\begin{aligned} A(\phi) &= \frac{24-N}{24\pi\epsilon}, \\ B(\phi) &= -\frac{1}{\pi\epsilon} + \frac{N}{4\pi\epsilon} \left[ (\Phi'(\phi))^2 - \Phi''(\phi) - 2\Phi'(\phi) \right], \\ C(\phi) &= \frac{N-1}{2\pi\epsilon}\, k. \end{aligned} \quad (11)$$

Let us renormalize the dilaton gravity. The action including counter terms is given in terms of bare quantities denoted by the suffix 0 as

$$\begin{aligned} S_0 &= S + S_{\rm counter} \quad (12) \\ &= \int d^d x \sqrt{-g_0} \left[ \frac{1}{16\pi G_0} R_0^{(d)} \, {\rm e}^{-2\phi_0} - \frac{1}{4\pi\alpha'_0} g_0^{\mu\nu} \partial_\mu X^i \partial_\nu X^j \bar{G}_{ij}(X) \, {\rm e}^{-2\Phi_0(\phi_0)} \right]. \end{aligned}$$

By defining the renormalized quantities as

$$\begin{aligned} \Phi_0(\phi_0) &= \Phi(\phi) + F(\phi) \qquad (F(0) = 0), \\ g_{0\,\mu\nu} &= g_{\mu\nu} \, {\rm e}^{-2\Lambda(\phi)} \qquad (\Lambda(0) = 0), \\ \phi_0 &= \phi + f(\phi) \qquad (f(0) = 0), \end{aligned} \quad (13)$$

the renormalization of the one-loop divergence can be achieved by requiring

$$\frac{1}{16\pi G_0} \, {\rm e}^{-2\phi - \epsilon\Lambda(\phi) - 2f(\phi)} = \mu^\epsilon \Big[ \frac{1}{16\pi G} \, {\rm e}^{-2\phi} - A(\phi) \Big],$$

$$
\begin{aligned}
&\frac{\epsilon+1}{16\pi G_0}\Big(4\Lambda' + \epsilon(\Lambda')^2 + 4f'\Lambda'\Big)(\phi)\,\mathrm{e}^{-2\phi-\epsilon\Lambda(\phi)-2f(\phi)} = -\mu^\epsilon B(\phi),\\
&\frac{1}{2\pi\alpha_0'}\,\mathrm{e}^{-2\Phi(\phi)-\epsilon\Lambda(\phi)-2F(\phi)} = \mu^\epsilon\Big[\frac{1}{2\pi\alpha'}\,\mathrm{e}^{-2\Phi(\phi)} - C(\phi)\Big].
\end{aligned}
\tag{14}
$$

Using the fact that $A(\phi)$, $C(\phi)$ are constant, $\Lambda, f, F = O(G,\alpha')$ at one-loop order, and neglecting higher order terms in $G$, $\alpha'$ we find the solution of these equations and obtain the relation between the bare and renormalized quantities as

$$
\begin{aligned}
\frac{1}{G_0} &= \mu^\epsilon\Big(\frac{1}{G} - 16\pi A\Big), \qquad \frac{1}{\alpha_0'} = \mu^\epsilon\Big(\frac{1}{\alpha'} - 2\pi C\Big),\\
\rho_0 &= \rho - \frac{4\pi G}{\epsilon+1}\int_0^\phi d\phi'\,\mathrm{e}^{2\phi'}B(\phi'),\\
\phi_0 &= \phi + 8\pi A G\left(\mathrm{e}^{2\phi}-1\right) + \frac{2\pi\epsilon G}{\epsilon+1}\int_0^\phi d\phi'\,\mathrm{e}^{2\phi'}B(\phi'),\\
\Phi_0(\phi_0) &= \Phi(\phi) + \pi\alpha' C\left(\mathrm{e}^{2\Phi(\phi)}-1\right) + \frac{2\pi\epsilon G}{\epsilon+1}\int_0^\phi d\phi'\,\mathrm{e}^{2\phi'}B(\phi')\\
&= \Phi(\phi_0) - 8\pi G A\Phi'(\phi_0)(\,\mathrm{e}^{2\phi_0}-1) + \pi\alpha' C\left(\mathrm{e}^{2\Phi(\phi_0)}-1\right)\\
&\quad -\frac{2\pi\epsilon G}{\epsilon+1}(\Phi'(\phi_0)-1)\int_0^{\phi_0} d\phi'\,\mathrm{e}^{2\phi'}B(\phi').
\end{aligned}
\tag{15}
$$

We find that the beta functions $\beta$ and the anomalous dimensions $\gamma$ are functions of $\phi$ in general

$$
\begin{aligned}
\beta_G &\equiv \mu\frac{\partial G}{\partial\mu} = \epsilon G - 16\pi\epsilon A G^2, \qquad \beta_{\alpha'} \equiv \mu\frac{\partial\alpha'}{\partial\mu} = \epsilon\alpha' - 2\pi\epsilon C\alpha'^2,\\
\beta_\Phi(\phi_0) &\equiv \mu\frac{\partial\Phi(\phi_0)}{\partial\mu}\\
&= 8\pi\epsilon A G\Phi'(\phi_0)\left(\mathrm{e}^{2\phi_0}-1\right) - \pi\epsilon\alpha' C\left(\mathrm{e}^{2\Phi(\phi_0)}-1\right)\\
&\quad +\frac{2\pi\epsilon^2 G}{\epsilon+1}\left(\Phi'(\phi_0)-1\right)\int_0^{\phi_0} d\phi'\,\mathrm{e}^{2\phi'}B(\phi'),\\
\gamma_\rho &\equiv \mu\frac{\partial\rho}{\partial\mu} = \frac{4\pi\epsilon G}{\epsilon+1}\int_0^\phi d\phi'\,\mathrm{e}^{2\phi'}B(\phi'),\\
\gamma_\phi &\equiv \mu\frac{\partial\phi}{\partial\mu} = -8\pi\epsilon A G\left(\mathrm{e}^{2\phi}-1\right) - \frac{2\pi\epsilon^2 G}{\epsilon+1}\int_0^\phi d\phi'\,\mathrm{e}^{2\phi'}B(\phi').
\end{aligned}
\tag{16}
$$

The beta function $\beta_G$ is identical to that of the Einstein gravity [3] as was reported in [7]. For $N < 24$ and $\epsilon > 0$, $G = 0$ is an infrared stable fixed point and

$G = G_*$ is an ultraviolet stable fixed point, where

$$G_* = \frac{3\epsilon}{2(24-N)}, \qquad \beta_G(G_*) = 0, \qquad \beta_G'(G_*) < 0. \tag{17}$$

The beta function $\beta_{\alpha'}$ coincides with that of a nonlinear $\sigma$-model with a maximally symmetric target space at one-loop order. For $\epsilon > 0$, the trivial fixed point $\alpha' = 0$ is infrared stable and the nontrivial fixed point $\alpha' = \alpha'_*$ is ultraviolet stable, irrespective of the sign of the target space curvature $k = \pm 1$

$$\alpha'_* = \frac{\epsilon}{(N-1)k}, \qquad \beta_{\alpha'}(\alpha'_*) = 0, \qquad \beta'_{\alpha'}(\alpha'_*) < 0. \tag{18}$$

The fixed point condition $\beta_\Phi(\phi) = 0$ for the dilaton coupling function $\Phi(\phi)$ is given by a functional equation

$$\begin{aligned} &\frac{24-N}{3} G(\mathrm{e}^{2\phi} - 1)\Phi'(\phi) - \frac{(N-1)k}{2}\alpha'\left(\mathrm{e}^{2\Phi(\phi)} - 1\right) \\ &\quad + \frac{2\pi\epsilon^2 G}{\epsilon+1}\left(\Phi'(\phi) - 1\right)\int_0^\phi d\phi' \,\mathrm{e}^{2\phi'}\left(-\frac{1}{\pi\epsilon} + \frac{N}{4\pi\epsilon}\left[(\Phi')^2 - \Phi'' - 2\Phi'\right](\phi')\right) = 0. \end{aligned} \tag{19}$$

An appropriate differentiation leads to a nonlinear differential equation

$$\begin{aligned} &\frac{(24-N)G}{3}\left[\left(\mathrm{e}^{2\phi} - 1\right)\Phi'' - 2\mathrm{e}^{2\phi}\Phi'\left(\Phi' - 1\right)\right] \\ &\quad - \frac{(N-1)k\alpha'}{2}\left[\left(\mathrm{e}^{2\Phi} - 1\right)\Phi'' - 2\mathrm{e}^{2\Phi}\Phi'\left(\Phi' - 1\right)\right] \\ &\quad + \frac{2\pi\epsilon^2 G}{\epsilon+1}\left(\Phi' - 1\right)^2 \mathrm{e}^{2\phi}\left(\frac{1}{\pi\epsilon} - \frac{N}{4\pi\epsilon}\left[(\Phi')^2 - \Phi'' - 2\Phi'\right]\right) = 0. \end{aligned} \tag{20}$$

This equation simplifies at the fixed point for $G = G_*$ and $\alpha' = \alpha'_*$. We find immediately that there is a simple solution

$$\Phi_*(\phi) = \phi, \tag{21}$$

although we are unable to find general solutions for this nonlinear equation. In particular, we find no consistent solution with $\Phi(\phi) = \lambda\phi$, $\lambda = O(\epsilon)$ which is similar to the solution for our previous case of free massless scalar fields for matter [7]. Since the above nonlinear equation is obtained by differentiating the fixed point condition,

it is a necessary but not a sufficient condition for the fixed point. Therefore we have checked that the above solution (21) really satisfies the fixed point condition $\beta_\Phi(\phi) = 0$ in Eq. (19).

Let us study the stability of this fixed point (17), (18) and (21).[§] We expand the beta functions around the fixed point

$$G = G_* + \delta G, \qquad \alpha' = \alpha'_* + \delta\alpha', \qquad \Phi(\phi) = \phi + \delta\Phi(\phi), \tag{22}$$

assuming the fluctuations $\delta G$, $\delta\alpha'$ and $\delta\Phi$ to be small. By examining the beta functions around the fixed point to first order in the fluctuations we find

$$\begin{aligned} \beta_G &= -\epsilon\delta G, \qquad \beta_{\alpha'} = -\epsilon\delta\alpha', \\ \beta_\Phi &= \left(\mathrm{e}^{2\phi} - 1\right)\left[\frac{24-N}{3}\delta G - \frac{(N-1)k}{2}\delta\alpha' + \frac{\epsilon}{2}\delta\Phi'\right] - \epsilon\,\mathrm{e}^{2\phi}\delta\Phi. \end{aligned} \tag{23}$$

We diagonalize these equations by introducing the following quantities

$$\begin{aligned} &\tilde{\Phi}(\phi) \equiv \Phi(\phi) + X(\phi)G + Y(\phi)\alpha', \\ &X(\phi) \equiv -\frac{1}{2G_*}\left(1 - \mathrm{e}^{2\phi}\right), \qquad Y(\phi) \equiv \frac{1}{2\alpha'_*}\left(1 - \mathrm{e}^{2\phi}\right). \end{aligned} \tag{24}$$

Then we consider the beta function for $\tilde{\Phi}$ and obtain

$$\beta_{\tilde{\Phi}} = -\left[\frac{\epsilon}{2}\left(1 - \mathrm{e}^{2\phi}\right)\frac{d}{d\phi} + \epsilon\,\mathrm{e}^{2\phi}\right]\delta\tilde{\Phi}. \tag{25}$$

If we define a variable $\psi$ to describe the weak coupling regions ($0 < \mathrm{e}^\phi < 1$) of the loop expansion parameter $\mathrm{e}^\phi$ [7],

$$\psi = \frac{1}{2}\ln(\mathrm{e}^{-2\phi} - 1), \qquad \begin{cases} \psi \to +\infty \Longleftrightarrow \phi \to -\infty, \\ \psi \to -\infty \Longleftrightarrow \phi \to 0, \end{cases} \tag{26}$$

Eq. (25) becomes

$$\beta_{\tilde{\Phi}} = \left[\frac{\epsilon}{2}\frac{d}{d\psi} - \frac{\epsilon}{\mathrm{e}^{2\psi}+1}\right]\delta\tilde{\Phi}. \tag{27}$$

[§] We would like to correct the stability argument in Ref. [7]. The correct form of the linearized beta functions Eq. (5.16) in Ref. [7] should be $\beta_G = -\epsilon\delta G$, $\beta_\Phi = -\frac{1}{2}\epsilon(1 - \mathrm{e}^{2\phi})\frac{d}{d\phi}\delta\Phi + O(\epsilon^2)$. Namely these should be diagonal from the beginning, so that the Eqs. (5.18), (5.20), (5.21) and (5.22) are valid for $\Phi$ instead of $\tilde{\Phi}$. Therefore we obtain the same conclusions on the stability and we need not introduce $\tilde{\Phi}$ defined in Eq. (5.17). Then the fine-tuning of $\Phi$ in Eq. (5.24) is simply $\Phi(\phi) = \lambda^*\phi$.

We find eigenfunctions of the differential operator with eigenvalues $\Lambda$ as

$$\delta\tilde{\Phi} = \frac{2}{\mathrm{e}^{2\psi}+1}\,\mathrm{e}^{\left(2+\frac{2}{\epsilon}\Lambda\right)\psi} = 2\,\mathrm{e}^{2\phi}\left(\mathrm{e}^{-2\phi}-1\right)^{(1+\frac{\Lambda}{\epsilon})}, \quad \beta_{\tilde{\Phi}} = \Lambda\delta\tilde{\Phi}. \tag{28}$$

The condition $\delta\tilde{\Phi}(\phi=0)=0$ requires $\Lambda > -\epsilon$. Therefore the fixed point is a saddle point: ultraviolet stable for direction $-\epsilon < \Lambda < 0$, and unstable for $\Lambda > 0$.

We can consider a renormalized theory with the gravitational coupling constant and the target space curvature near the fixed point $G_*$, $\alpha'_*$, as long as we fine tune the dilaton coupling function $\tilde{\Phi}$ to be precisely at the fixed point $\delta\tilde{\Phi}=0$

$$\begin{aligned}\Phi(\phi) &= \phi - X(\phi)(G-G_*) - Y(\phi)(\alpha'-\alpha'_*) \\ &= \phi + \frac{1}{2}\left(\frac{G}{G_*}-\frac{\alpha'}{\alpha'_*}\right)\left(1-\mathrm{e}^{2\phi}\right).\end{aligned} \tag{29}$$

After fine-tuning of the dilaton coupling function, we find the dilaton gravity theory around the fixed points for the gravitational constant $G$ and the curvature of the target space $\alpha'$

$$\begin{aligned}S = \mu^\epsilon \int d^dx\sqrt{-g}\,\mathrm{e}^{-2\phi}\Big[&\frac{1}{16\pi G}R^{(d)} \\ &-\frac{1}{4\pi\alpha'}\,\mathrm{e}^{-\left(\frac{G}{G_*}-\frac{\alpha'}{\alpha'_*}\right)(1-\mathrm{e}^{2\phi})} g^{\mu\nu}\partial_\mu X^i\partial_\nu X^j \bar{G}_{ij}(X)\Big].\end{aligned} \tag{30}$$

Let us examine the meaning of the fixed point theory for the dilaton gravity coupled to the maximally symmetric nonlinear sigma model. If we tentatively allow a singular Weyl rescaling $g_{\mu\nu} \to g_{\mu\nu}\,\mathrm{e}^{\frac{4}{\epsilon}\phi}$, we obtain

$$\begin{aligned}S = \mu^\epsilon \int d^dx\sqrt{-g}\Big[&\frac{1}{16\pi G_*}\left(R^{(d)} - \frac{4(1+\epsilon)}{\epsilon}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) \\ &-\frac{1}{4\pi\alpha'_*}g^{\mu\nu}\partial_\mu X^i\partial_\nu X^j\bar{G}_{ij}(X)\Big].\end{aligned} \tag{31}$$

This is the Einstein gravity coupled to a free scalar $\phi$ and to scalar fields $X^i$ which form a maximally symmetric target space. The dilaton field can be put into a free scalar field embedded into the Einstein gravity as long as we stay away from two dimensions. Let us emphasize, however, that the $(2+\epsilon)$-dimensional approach does not permit such a singular local Weyl rescaling and the fixed point theory cannot be regarded as a truly free dilaton theory.

We finally clarify a possible relation between the model studied in this paper and the model in our previous paper [7]. At the tree level, the theory of the dilaton gravity coupled to free scalar fields can be reproduced from the dilaton gravity coupled to matter fields with a maximally symmetric target space by taking the limit $\alpha' \to 0$. This can be easily understood from the fact that the parameter $\alpha'$ is the curvature of the target space. The fixed point theory, however, is not connected to that of the dilaton gravity coupled to free matter fields. Let us compare the fixed point corresponding to the nonlinear sigma model

$$G_* = \frac{3\epsilon}{2(24-N)}, \qquad \alpha'_* = \frac{\epsilon}{(N-1)k}, \qquad \Phi_*(\phi) = \phi, \tag{32}$$

and the fixed point corresponding to the previous model with free massless scalar matter fields [7]

$$G_* = \frac{3\epsilon}{2(24-N)}, \qquad \alpha'_* = 0, \qquad \Phi_*(\phi) = -\frac{3\epsilon}{24-N}\phi. \tag{33}$$

Actually these are two different fixed points in the coupling constant space. In the direction of $\alpha'$, the former fixed point is ultraviolet stable. On the other hand, the latter is ultraviolet unstable and a fine-tuning is necessary. This fact is the origin of the different forms of the dilaton-matter coupling function $\Phi_*(\phi)$. Even using the freedom of field redefinitions, we cannot bring our new fixed point theory to the previous one.

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